JOURNAL OF COMBINATORIAL THEORY, Series B **70**, 184–196 (1997)
ARTICLE NO. TB971749

Two Chromatic Polynomial Conjectures

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Received May 23, 1996

DEDICATED TO PROFESSOR W. T. TUTTE ON THE OCCASION OF HIS EIGHTIETH BIRTHDAY

Let $P(\lambda)$ be the chromatic polynomial of a graph. We show that $P(5)^{-1}\, P(6)^2\, P(7)^{-1}$ can be arbitrarily small, disproving a conjecture of Welsh (and of Brenti,

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and also disproving several other conjectures of Brenti. Secondly, we prove that if the graph has n vertices, then

$$P(n)\, P(n-1)^{-1} \geqslant 2.718253,$$

approaching a conjecture of Bartels and Welsh that $P(n)\, P(n-1)^{-1} \geqslant e$ (e is 2.718281 ...). © 1997 Academic Press

1. INTRODUCTION

Let G be a graph (in this paper, all graphs are finite and simple) and for $\lambda \geqslant 1$, let $P(\lambda)$ denote the number of λ-colourings of G. (A *λ-colouring* means a map ϕ: $V(G) \to \{1, ..., \lambda\}$ such that $\phi(u) \neq \phi(v)$ whenever u and v are adjacent vertices.) We are concerned with two conjectures about $P(\lambda)$. The first is the following:

(1.1) CONJECTURE. *For all integers $\lambda \geqslant 1$, $P(\lambda)^2 \geqslant P(\lambda-1)\, P(\lambda+1)$.*

This was proposed by Welsh (private communication) in the early 1970's, and later, independently, by Brenti [2]. We shall show that (1.1) is false; indeed, in Section 2 we exhibit graphs with $P(5)^{-1}\, P(6)^2\, P(7)^{-1}$ arbitrarily small.

The second conjecture, due to Bartels and Welsh [1], is the following ($e = 2.7182818...$ is the base of natural logarithms):

(1.2) CONJECTURE. *If $|V(G)| = n$ then $P(n)\, P(n-1)^{-1} \geqslant e$.*

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This remains open, but we prove that $P(n)P(n-1)^{-1} \geqslant \frac{685}{252}$ $(=2.7182539...)$. This is proved in Section 3.

2. THE COUNTEREXAMPLE

Let $\mathscr{E}$ be

$$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{3, 5\}, \{1, 6\}, \{2, 6\}\},$$

and let $A_1, ..., A_6$ be mutually disjoint sets each of cardinality n. Let G have vertex set $A_1 \cup \cdots \cup A_6$, and for distinct $u, v \in V(G)$ with $u \in A_i$ and $v \in A_j$ say, let u, v be adjacent if and only if $\{i, j\} \in \mathscr{E}$.

(2.1) *With G as above*, $P(5) \geqslant 27^n$, $P(7) \geqslant 216^n$, *and*

$$P(6) \leqslant 1080 \cdot 72^n + 210 \cdot 64^n + 360 \cdot 48^n + 360 \cdot 36^n + 90 \cdot 16^n.$$

Proof. Let ϕ: $V(G) \to \{1, ..., 5\}$ satisfy, for $1 \leqslant i \leqslant 6$ and all $v \in A_i$, $\phi(v) = i$ if $i \leqslant 3$ and $\phi(v) \in \{i-3, 4, 5\}$ if $i \geqslant 4$. Each such ϕ is a 5-colouring of G, and since there are 27^n such maps ϕ it follows that $P(5) \geqslant 27^n$.

Now let ϕ: $V(G) \to \{1, ..., 7\}$ satisfy, for $1 \leqslant i \leqslant 6$ and all $v \in A_i$, $\phi(v) \in \{i, i+3\}$ if $i \leqslant 3$, and $\phi(v) \in \{i-3, i, 7\}$ if $i \geqslant 4$. Each such ϕ is a 7-colouring, so $P(7) \geqslant 216^n$.

Now let us get the upper bound on $P(6)$. A *pattern* is a map ψ with domain $\{1, ..., 6\}$ such that

(i) for $1 \leqslant i \leqslant 6$, $\psi(i) \subseteq \{1, ..., 6\}$ and $\psi(i) \neq \varnothing$

(ii) $\psi(i) \cap \psi(j) = \varnothing$ for all $\{i, j\} \in \mathscr{E}$.

If ψ_1, ψ_2 are patterns, we say $\psi_1 \leqslant \psi_2$ if $\psi_1(i) \subseteq \psi_2(i)$ for $1 \leqslant i \leqslant 6$; and ψ_1 is *maximal* if there is no pattern $\psi_2 \neq \psi_1$ with $\psi_1 \leqslant \psi_2$. A 6-colouring ϕ of G *obeys* a pattern ψ if $\phi(v) \in \psi(i)$ for all i $(1 \leqslant i \leqslant 6)$ and all $v \in A_i$. Every 6-colouring ϕ obeys the pattern ψ defined by

$$\psi(i) = \{\phi(v): v \in A_i\} \qquad (1 \leqslant i \leqslant 6),$$

and consequently, ϕ also obeys some maximal pattern.

If ψ is a pattern, its *worth* $w(\psi)$ is $\prod_{i=1, ..., 6} |\psi(i)|$. The number of 6-colourings obeying a given pattern ψ is $w(\psi)^n$, and so $P(6) \leqslant \sum w(\psi)^n$, summed over all maximal patterns ψ. Now easy case analysis shows that there are 1080 maximal patterns of worth 72, 210 of worth 64, 360 of worth 48, 360 of worth 36, 90 of worth 16, and no others. The result follows. ∎

From (2.1) we see (by taking n large) that $P(5)^{-1} P(6)^2 P(7)^{-1}$ may be less than 1, and indeed may be arbitrarily small.

3. THE SECOND CONJECTURE

Now we prove a positive result, the following:

(3.1) *Let* $|V(G)| = n \geqslant 1$. *Then* $P(n) \geqslant (685/252)\, P(n-1)$.

Throughout this section, let us fix G and n with $|V(G)| = n \geqslant 4$. (It is easy to verify (3.1) for $n \leqslant 3$.) We denote by $\mathscr{A}$ the set of all sets $\{A_1, ..., A_k\}$ such that each A_i is a non-null stable subset of $V(G)$, and the sets $A_1, ..., A_k$ are pairwise disjoint and have union $V(G)$. The *score* of $\{A_1, ..., A_k\}$ is the $(n+1)$-tuple $(s_0, s_1, ..., s_n)$, where $s_0 = n-k$ and, for $1 \leqslant i \leqslant n$, s_i is the number of $A_1, ..., A_k$ that have cardinality i.

Let $\mathscr{S}$ denote the set of all $(n+1)$-tuples $(s_0, ..., s_n)$ of integers satisfying

$$\sum_{0 \leqslant i \leqslant n} s_i = \sum_{0 \leqslant i \leqslant n} i s_i = n.$$

Then the score of every member of $\mathscr{A}$ belongs to $\mathscr{S}$, as is easily seen.

We use vector notation for members of $\mathscr{S}$; thus, $\mathbf{s}$ means $(s_0, s_1, ..., s_n)$ and $\mathbf{s} + \mathbf{t}$ means $(s_0 + t_0, ..., s_n + t_n)$. For $\mathbf{s} \in \mathscr{S}$, let $\mathscr{A}(\mathbf{s})$ denote the set of all members of $\mathscr{A}$ with score $\mathbf{s}$. Thus, $\mathscr{A}(\mathbf{s})$ is null unless $s_0, ..., s_n$ are all non-negative.

(3.2) *For every integer* $\lambda \geqslant 1$,

$$P(\lambda) = \sum_{\mathbf{s} \in \mathscr{S}} \lambda(\lambda-1) \cdots (\lambda - n + s_0 + 1)\, |\mathscr{A}(\mathbf{s})|.$$

Proof. Every $\{A_1, ..., A_k\} \in \mathscr{A}$ yields $\lambda(\lambda-1) \cdots (\lambda-k+1)$ colourings in the obvious way, and every colouring arises from a unique member of $\mathscr{A}$. Since every member of $\mathscr{A}$ belongs to a unique $\mathscr{A}(\mathbf{s})$, the result follows. ▮

In particular, for $\mathbf{s} \in \mathscr{S}$ let

$$M(\mathbf{s}) = (n-1)(n-2) \cdots s_0\, |\mathscr{A}(\mathbf{s})|$$

$$N(\mathbf{s}) = n(n-1) \cdots (s_0+1)\, |\mathscr{A}(\mathbf{s})|.$$

From (3.2) we have

(3.3) $P(n-1) = \sum_{\mathbf{s} \in \mathscr{S}} M(\mathbf{s})$, *and* $P(n) = \sum_{\mathbf{s} \in \mathscr{S}} N(\mathbf{s})$.

We therefore wish to show that

$$\sum_{\mathbf{s} \in \mathscr{S}} \left(\frac{252}{685} N(\mathbf{s}) - M(\mathbf{s}) \right) \geqslant 0.$$

It is not true that $(252/685)\, N(\mathbf{s}) - M(\mathbf{s}) \geqslant 0$ for all $\mathbf{s} \in \mathscr{S}$. It is true, however, that for every $\mathbf{s} \in \mathscr{S}$ there exists $\mathbf{t} \in \mathscr{S}$ differing from $\mathbf{s}$ only a little, with $(252/685)\, N(\mathbf{s}) \geqslant M(\mathbf{t})$. (This is a consequence of (3.5) below.) By itself this does not prove (3.1); we must then "smooth out" the dependence of $\mathbf{t}$ on $\mathbf{s}$, so that $\mathbf{t}$ ranges uniformly over $\mathscr{S}$ when we sum over all $\mathbf{s} \in \mathscr{S}$. To achieve this we take an appropriate linear combination of the various inequalities of (3.5).

We evidently have

(3.4) *For* $\mathbf{s} \in \mathscr{S}$, $nM(\mathbf{s}) = s_0 N(\mathbf{s})$.

Less trivially, we have the following. For $1 \leqslant i \leqslant n$, let $\delta_i = (d_0, d_1, ..., d_n)$, where $d_0, ..., d_n = 0$ except that $d_i = 1$ and $d_0 = -1$.

(3.5) *Let* $\mathbf{s} \in \mathscr{S}$. *Then*

(i) *for* $1 \leqslant i \leqslant \frac{1}{2}n$,

$$n \binom{2i}{i} s_{2i} M(\mathbf{s}) \leqslant (s_i + 1)(s_i + 2)\, N(\mathbf{s} + 2\delta_i - \delta_{2i});$$

(ii) *for* $1 \leqslant i, j \leqslant n$ *with* $i \neq j$ *and* $i + j \leqslant n$,

$$n \binom{i+j}{i} s_{i+j} M(\mathbf{s}) \leqslant (s_i + 1)(s_j + 1)\, N(\mathbf{s} + \delta_i + \delta_j - \delta_{i+j}).$$

Proof. We prove only the second statement; the first is similar. Let $i, j \geqslant 1$ with $i \neq j$ and $i + j \leqslant n$. We may assume that $s_{i+j} \neq 0$, for otherwise the claim is trivial. Let $\mathbf{s}' = \mathbf{s} + \delta_i + \delta_j - \delta_{i+j}$. Then $\mathbf{s}' \in \mathscr{S}$. Let $\{A_1, ..., A_k\} \in \mathscr{A}(\mathbf{s})$ and $\{A'_1, ..., A'_{k'}\} \in \mathscr{A}(\mathbf{s}')$ (thus, $k = n - s_0 = n - s'_0 - 1 = k' - 1$). We say they are *related* if there are distinct p, q with $1 \leqslant p, q \leqslant k'$ and r with $1 \leqslant r \leqslant k$, so that $A'_p \cup A'_q = A_r$, $|A'_p| = i$, $|A'_q| = j$, and

$$\{A_1, ..., A_k\} - \{A_r\} = \{A'_1, ..., A'_{k'}\} - \{A'_p, A'_q\}.$$

Each member of $\mathscr{A}(\mathbf{s}')$ is related to at most $s'_i s'_j$ members of $\mathscr{A}(\mathbf{s})$ (equality may not hold since the union of two stable sets need not be stable), and each member of $\mathscr{A}(\mathbf{s})$ is related to exactly $\binom{i+j}{i} s_{i+j}$ members of $\mathscr{A}(\mathbf{s}')$. Thus,

$$\binom{i+j}{i} s_{i+j}\, |\mathscr{A}(\mathbf{s})| \leqslant s'_i s'_j\, |\mathscr{A}(\mathbf{s}')|.$$

Consequently,

$$
\begin{aligned}
n\binom{i+j}{i} s_{i+j} M(\mathbf{s}) &= n\binom{i+j}{i} s_{i+j}(n-1)(n-2)\cdots s_0\, |\mathscr{A}(\mathbf{s})| \\
&\leqslant s'_i s'_j n(n-1)\cdots s_0\, |\mathscr{A}(\mathbf{s}')| \\
&= s'_i s'_j n(n-1)\cdots (s'_0+1)\, |\mathscr{A}(\mathbf{s}')| \\
&= s'_i s'_j N(\mathbf{s}'),
\end{aligned}
$$

as required. ▮

Let $\mathscr{T}$ be the set of all triples (x_0, x_1, x_2) of non-negative integers so that $x_0+x_1+x_2 \leqslant n$ and $3x_0+2x_1+x_2 \geqslant 2n$. We observe first that

(3.6) *If* $\mathbf{s} \in \mathscr{S}$ *and* $s_0, s_1, ..., s_n \geqslant 0$ *then* $(s_0, s_1, s_2) \in \mathscr{T}$.

Proof. Since

$$s_0+s_1+s_2 \leqslant \sum_{0 \leqslant i \leqslant n} s_i = n$$

and

$$3s_0+2s_1+s_2 \geqslant \sum_{0 \leqslant i \leqslant n} (3-i) s_i = 3n-n = 2n,$$

the result follows. ▮

For any triple of integers (x_0, x_1, x_2), let

$$
\begin{aligned}
\mathscr{S}(x_0, x_1, x_2) &= \{\mathbf{s} \in \mathscr{S}: s_0 = x_0, s_1 = x_1, s_2 = x_2\}, \\
M(x_0, x_1, x_2) &= \sum_{\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)} M(\mathbf{s}),
\end{aligned}
$$

and define $N(x_0, x_1, x_2)$ similarly. From (3.4), (3.5) and (3.6) we deduce:

(3.7) *Let* $(x_0, x_1, x_2) \in \mathscr{T}$. *Then the following four statements hold*:

(i) $2nx_2 M(x_0, x_1, x_2) \leqslant (x_1+1)(x_1+2)\, N(x_0-1, x_1+2, x_2-1)$

(ii) $n(n-x_1-2x_2)\, M(x_0, x_1, x_2)$

$$\leqslant (x_1+1)(x_2+1)\, N(x_0-1, x_1+1, x_2+1) + (x_1+1)(n-x_0-x_1-x_2)\, N(x_0-1, x_1+1, x_2)$$

(iii) $5n(3x_0+2x_1+x_2-2n)\, M(x_0, x_1, x_2)$

$$\leqslant \tfrac{5}{6}(x_2+1)(x_2+2)\, N(x_0-1, x_1, x_2+2) + (x_2+1)(n-x_0-x_1-x_2)\, N(x_0-1, x_1, x_2+1)$$

(iv) $nM(x_0, x_1, x_2) = x_0 N(x_0, x_1, x_2)$.

Proof. To prove the first statement, let $\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)$. By (3.5)(i), taking $i = 1$, we have

$$2nx_2 M(\mathbf{s}) \leqslant (x_1 + 1)(x_1 + 2)\, N(\mathbf{s} + 2\delta_1 - \delta_2).$$

Summing over all $\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)$ we obtain (by (3.6))

$$\begin{aligned} 2nx_2 M(x_0, x_1, x_2) &= \sum_{\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)} 2nx_2 M(\mathbf{s}) \\ &\leqslant \sum_{\mathbf{t} \in \mathscr{S}(x_0 - 1, x_1 + 2, x_2 - 1)} (x_1 + 1)(x_1 + 2)\, N(\mathbf{t}) \\ &= (x_1 + 1)(x_1 + 2)\, N(x_0 - 1, x_1 + 2, x_2 - 1). \end{aligned}$$

This proves the first statement.

For the second, we have from (3.5)(ii) that for all $\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)$ and $j \geqslant 2$,

$$n(j + 1)\, s_{j+1} M(\mathbf{s}) \leqslant (x_1 + 1)(s_j + 1)\, N(\mathbf{s} + \delta_1 + \delta_j - \delta_{j-1}).$$

Summing over all j with $2 \leqslant j < n$ and all $\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)$, and using that

$$\sum_{2 \leqslant j < n} (j + 1) s_{j+1} = n - s_1 - 2s_2 = n - x_1 - 2x_2,$$

we obtain

$$\begin{aligned} &n(n - x_1 - 2x_2)\, M(x_0, x_1, x_2) \\ &\quad \leqslant \sum_{\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)} \sum_{2 \leqslant j < n} (x_1 + 1)(s_j + 1)\, N(\mathbf{s} + \delta_1 + \delta_j - \delta_{j+1}) \\ &\quad = \sum_{\mathbf{t} \in \mathscr{S}(x_0 - 1, x_1 + 1, x_2 + 1)} (x_1 + 1)(x_2 + 1)\, N(\mathbf{t}) \\ &\quad\quad + \sum_{\mathbf{t} \in \mathscr{S}(x_0 - 1, x_1 + 1, x_2)} \sum_{3 \leqslant j < n} (x_1 + 1)\, t_j N(\mathbf{t}) \\ &\quad = (x_1 + 1)(x_2 + 1)\, N(x_0 - 1, x_1 + 1, x_2 + 1) \\ &\quad\quad + (x_1 + 1)(n - x_0 - x_1 - x_2)\, N(x_0 - 1, x_1 + 1, x_2) \end{aligned}$$

(since $t_1 > 0$ and hence $t_n = 0$ and $\sum_{3 \leqslant j < n} t_j = n - x_0 - x_1 - x_2$ for all $\mathbf{t} \in \mathscr{S}(x_0 - 1,\ x_1 + 1,\ x_2)$). This proves the second statement.

For the third, from (3.5)(i) (taking $i = 2$) we have that for every $\mathbf{s} \in \mathscr{S}(x_0, x_1, x_2)$,

$$6ns_4 M(\mathbf{s}) \leqslant (x_2 + 1)(x_2 + 2)\, N(\mathbf{s} + 2\delta_2 - \delta_4).$$

Also, from (3.5)(ii), taking $i=2$ and summing over all j with $3\leqslant j\leqslant n-2$, we have (since $\frac{1}{2}(j+1)(j+2)\geqslant 5(j-1)$ for $j\geqslant 3$) that for all $\mathbf{s}\in\mathscr{S}(x_0, x_1, x_2)$,

$$\sum_{3\leqslant j\leqslant n-2} 5n(j-1)\, s_{j+2}M(\mathbf{s}) \leqslant \sum_{3\leqslant j\leqslant n-2} (x_2+1)(s_j+1)\, N(\mathbf{s}+\delta_2+\delta_j-\delta_{j+2}).$$

Adding this and $\frac{5}{6}$ times the previous inequality, we obtain

$$\begin{aligned}\sum_{2\leqslant j\leqslant n-2} 5n(j-1)\, s_{j+2}M(\mathbf{s}) &\leqslant \tfrac{5}{6}(x_2+1)(x_2+2)\, N(\mathbf{s}+2\delta_2-\delta_4)\\ &\quad+\sum_{3\leqslant j\leqslant n-2} (x_2+1)(s_j+1)\, N(\mathbf{s}+\delta_2+\delta_j-\delta_{j+2}).\end{aligned}$$

Now

$$\begin{aligned}\sum_{2\leqslant j\leqslant n-2} (j-1)s_{j+2} &= \sum_{4\leqslant j\leqslant n} js_j - \sum_{4\leqslant j\leqslant n} 3s_j\\ &=(n-s_1-2s_2-3s_3)-3(n-s_0-s_1-s_2-s_3)\\ &=3x_0+2x_1+x_2-2n;\end{aligned}$$

so summing the inequality over all $\mathbf{s}\in\mathscr{S}(x_0, x_1, x_2)$ yields

$$\begin{aligned}5n(3x_0+2x_1+x_2-2n)\, M(x_0, x_1, x_2) &\leqslant \tfrac{5}{6}(x_2+1)(x_2+2) \sum_{\mathbf{t}\in\mathscr{S}(x_0-1,\, x_1,\, x_2+2)} N(\mathbf{t})\\ &\quad+(x_2+1) \sum_{\mathbf{t}\in\mathscr{S}(x_0-1,\, x_1,\, x_2+1)}\ \sum_{3\leqslant j\leqslant n-2} t_jN(\mathbf{t})\\ &=\tfrac{5}{6}(x_2+1)(x_2+2)\, N(x_0-1, x_1, x_2+2)\\ &\quad+(x_2+1)(n-x_0-x_1-x_2)\, N(x_0-1, x_1, x_2+1)\end{aligned}$$

(since $t_2>0$ and hence $t_n=t_{n-1}=0$ and $\sum_{3\leqslant j\leqslant n-2} t_j=n-x_0-x_1-x_2$ for all $\mathbf{t}\in\mathscr{S}(x_0-1, x_1, x_2+1)$). This proves the third statement. The fourth follows easily from (3.4). ∎

Throughout the remainder of this section, let $\theta=\frac{685}{252}$, for all integers x let

$$\rho(x)=\theta^2\frac{x^2}{n^2}+\theta(12-\theta)\frac{x}{n}+72-11\theta,$$

and for all triples of integers $\mathbf{x}=(x_0, x_1, x_2)$ let

$$\alpha_1(\mathbf{x})=12\theta^2-110\theta+424$$

$$\alpha_2(\mathbf{x})=12\theta^2-110\theta+424+\theta\rho(x_1)+2\theta^2\frac{x_2}{n}$$

$$\alpha_3(\mathbf{x})=24\theta$$

$$\alpha_4(\mathbf{x})=360n+24\theta x_2+\theta x_1\ \rho(x_1)+2\theta^2\frac{x_1x_2}{n}+(12\theta^2-110\theta+424)x_1$$

$$\beta(\mathbf{x})=2nx_2\alpha_1(\mathbf{x})+n(n-x_1-2x_2)\,\alpha_2(\mathbf{x})$$
$$+5n(3x_0+2x_1+x_2-2n)\,\alpha_3(\mathbf{x})+n\alpha_4(\mathbf{x}).$$

If x_0, x_1, x_2 are all non-negative, it follows that $\beta(\mathbf{x})>0$; for $1\leqslant i\leqslant 4$ define $\gamma_i(\mathbf{x})=\alpha_i(\mathbf{x})\,\beta(\mathbf{x})^{-1}$ if x_0, x_1, x_2 are all non-negative, and $\gamma_i(\mathbf{x})=0$ otherwise.

We need one more lemma before the main proof, the following set of numerical inequalities:

(3.8) *For all* $\mathbf{x}\in\mathscr{T}$, *the following six inequalities hold* (*we recall that* $n\geqslant 4$):

(i) $(x_1-1)\,\gamma_1(x_0+1, x_1-2, x_2+1)\leqslant x_1\gamma_1(\mathbf{x})$

(ii) $\gamma_2(x_0+1, x_1-1, x_2-1)\leqslant\gamma_2(\mathbf{x})$

(iii) $\gamma_2(x_0+1, x_1-1, x_2)\leqslant\gamma_2(\mathbf{x})$

(iv) $\gamma_3(x_0+1, x_1, x_2-2)\leqslant\gamma_3(\mathbf{x})$

(v) $\gamma_3(x_0+1, x_1, x_2-1)\leqslant\gamma_3(\mathbf{x})$

(vi) $x_1^2\gamma_1(\mathbf{x})+x_1(n-x_0-x_1)\,\gamma_2(\mathbf{x})+x_2(n-x_0-x_1-\frac{1}{6}x_2)\,\gamma_3(\mathbf{x})+x_0\gamma_4(\mathbf{x})\leqslant\theta^{-1}$.

Proof. For any choice of integers ε_1 and ε_2, define

$$\delta(\varepsilon_1, \varepsilon_2)=\beta(x_0+1, x_1-\varepsilon_1, x_2-\varepsilon_2)-\beta(\mathbf{x}).$$

Then we have:

$$\delta(\varepsilon_1, \varepsilon_2)=24\theta n(15-10\varepsilon_1-6\varepsilon_2)+2\theta n\varepsilon_2\ \rho(x_1)+4\theta^2\varepsilon_2x_2$$
$$-\theta^2(n+2\varepsilon_2-2x_2)\left(\frac{\theta}{n}(2x_1\varepsilon_1-\varepsilon_1^2)+(12-\theta)\varepsilon_1+2\varepsilon_2\right).$$

(We shall leave the reader to verify this and similar statements.) We first prove statements (ii)–(v), since they are easier. For (ii), the above implies

$$\delta(1,1)=-24\theta n+2\theta n\rho(x_1)+4\theta^2 x_2-\theta^2(n+2-2x_2)\left(\frac{\theta}{n}(2x_1-1)+14-\theta\right).$$

Since $x_2\geqslant 0$, $\rho(x_1)\geqslant 72-11\theta+\theta(12-\theta)(x_1/n)$ and $n+2-2x_2\leqslant\frac{3}{2}n$, it follows that

$$\delta(1,1)\geqslant -24\theta n+2\theta n\left(72-11\theta+\theta(12-\theta)\frac{x_1}{n}\right)-\frac{3}{2}\theta^2 n\left(2\theta\frac{x_1}{n}+14-\theta\right)$$
$$=\theta n\left(120-43\theta+\frac{3}{2}\theta^2\right)+\theta^2 x_1(24-5\theta)\geqslant 0.$$

Thus $\beta(x_0+1,x_1-1,x_2-1)\geqslant\beta(\mathbf{x})>0$, and since $\alpha_2(x_0+1,x_1-1,x_2-1)\leqslant\alpha_2(\mathbf{x})$, this proves (ii). Statements (iii)–(v) are proved similarly; we omit the details (in fact, all three are easier than (ii)).

For (i), we must show that $x_1\delta(2,-1)+\beta(\mathbf{x})\geqslant 0$. Since $3x_0+2x_1+x_2-2n\geqslant 0$, it suffices to show that $F(x_1,x_2)\geqslant 0$, where

$$F(x_1,x_2)=x_1\delta(2,-1)+\beta(x_0,x_1,x_2)+120\theta n(2n-3x_0-2x_1-x_2).$$

(Note that $F(x_1,x_2)$ is indeed independent of x_0.) Now since $\mathbf{x}\in\mathscr{T}$ it follows that

$$x_1+2x_2=3(x_0+x_1+x_2)-(3x_0+2x_1+x_2)\leqslant 3n-2n=n.$$

It suffices therefore to verify that $F(x_1,x_2)\geqslant 0$ for all real numbers x_1, x_2 such that $0\leqslant x_1\leqslant n$ and $0\leqslant x_2\leqslant\frac{1}{2}(n-x_1)$. For fixed x_1, $F(x_1,x_2)$ is quadratic in x_2, with the coefficient of x_2^2 negative; and so to show that $F(x_1,x_2)\geqslant 0$ for all x_2 with $0\leqslant x_2\leqslant\frac{1}{2}(n-x_1)$ it suffices to show it when $x_2=0$ and when $x_2=\frac{1}{2}(n-x_1)$. But $F(x_1,0)$ and $F(x_1,\frac{1}{2}(n-x_1))$ are both cubic in x_1, with all coefficients negative except the constant term, and hence are minimized when $x_1=n$, when they are equal; and $F(n,0)\geqslant 0$. Hence (i) follows.

Finally, let us prove (vi). After substituting and simplifying, the formula (vi) that we need to prove becomes

$$\frac{\theta^3}{n^2}x_1^4-(2\theta^3-12\theta^2)\frac{x_1^3}{n}+(\theta^3-22\theta^2+72\theta)x_1^2-(2\theta^2-50\theta+184)nx_1$$
$$-\left(\theta-278+\frac{784}{\theta}\right)n^2\geqslant 0.$$

This expression factors (using the value $\frac{685}{252}$ of θ to adjust the constant term—this is the only place in the proof of (3.1) where the exact value of θ is used) as

$$\theta\left(x_1-\frac{n}{\theta}\right)^2\left(\frac{\theta^2}{n^2}x_1^2+(14-2\theta)\,\theta\,\frac{x_1}{n}+\theta^2-26\theta+99\right);$$

and this is non-negative, for all $x_1 \geqslant 0$. ∎

Proof of (3.1). Multiplying the four statements of (3.7) by $\gamma_1(\mathbf{x})$, ..., $\gamma_4(\mathbf{x})$ respectively, and adding, we deduce that for all $\mathbf{x} \in \mathscr{T}$,

$$\begin{aligned}
&(2nx_2\gamma_1(\mathbf{x})+n(n-x_1-2x_2)\gamma_2(\mathbf{x})\\
&\qquad+5n(3x_0+2x_1+x_2-2n)\gamma_3(\mathbf{x})+n\gamma_4(\mathbf{x}))\,M(\mathbf{x})\\
&\quad\leqslant(x_1+1)(x_1+2)\,\gamma_1(\mathbf{x})\,N(x_0-1,x_1+2,x_2-1)\\
&\qquad+(x_1+1)(x_2+1)\,\gamma_2(\mathbf{x})\,N(x_0-1,x_1+1,x_2+1)\\
&\qquad+(x_1+1)(n-x_0-x_1-x_2)\,\gamma_2(\mathbf{x})\,N(x_0-1,x_1+1,x_2)\\
&\qquad+\tfrac{5}{6}(x_2+1)(x_2+2)\,\gamma_3(\mathbf{x})\,N(x_0-1,x_1,x_2+2)\\
&\qquad+(x_2+1)(n-x_0-x_1-x_2)\,\gamma_3(\mathbf{x})\,N(x_0-1,x_1,x_2+1)\\
&\qquad+x_0\gamma_4(\mathbf{x})\,N(x_0,x_1,x_2).
\end{aligned}$$

Now the quantity on the left side of this inequality equals $M(\mathbf{x})$; so, summing over all $\mathbf{x} \in \mathscr{T}$, we deduce that

$$\sum_{\mathbf{x}\in\mathscr{T}} M(\mathbf{x}) \leqslant \sum_{\mathbf{x}\in\mathscr{T}} N(\mathbf{x})(Z_1(\mathbf{x})+\cdots+Z_6(\mathbf{x})),$$

where (by (3.8)(i) $\cdots$ (v))

$$\begin{aligned}
Z_1(\mathbf{x})&=x_1(x_1-1)\,\gamma_1(x_0+1,x_1-2,x_2+1)\leqslant x_1^2\gamma_1(\mathbf{x})\\
Z_2(\mathbf{x})&=x_1x_2\gamma_2(x_0+1,x_1-1,x_2-1)\leqslant x_1x_2\gamma_2(\mathbf{x})\\
Z_3(\mathbf{x})&=x_1(n-x_0-x_1-x_2)\,\gamma_2(x_0+1,x_1-1,x_2)\\
&\qquad\leqslant x_1(n-x_0-x_1-x_2)\,\gamma_2(\mathbf{x})\\
Z_4(\mathbf{x})&=\tfrac{5}{6}x_2(x_2-1)\,\gamma_3(x_0+1,x_1,x_2-2)\leqslant\tfrac{5}{6}x_2^2\gamma_3(\mathbf{x})\\
Z_5(\mathbf{x})&=x_2(n-x_0-x_1-x_2)\,\gamma_3(x_0+1,x_1,x_2-1)\\
&\qquad\leqslant x_2(n-x_0-x_1-x_2)\,\gamma_3(\mathbf{x})\\
Z_6(\mathbf{x})&=x_0\gamma_4(\mathbf{x}).
\end{aligned}$$

Hence, by (3.8)(vi), $Z_1(\mathbf{x}) + \cdots + Z_6(\mathbf{x}) \leqslant \theta^{-1}$; and so

$$\sum_{\mathbf{x} \in \mathscr{T}} M(\mathbf{x}) \leqslant \sum_{\mathbf{x} \in \mathscr{T}} N(\mathbf{x}) \theta^{-1}$$

and the result follows from (3.3), as required. ∎

4. REMARKS

There are a number of interrelated conjectures about the chromatic polynomial $P(\lambda)$ (see for instance [2]). The most well-known is probably Read's conjecture [4]. It is known that $P(\lambda)$ is a polynomial in λ, say $(-1)^n \sum_{0 \leqslant i \leqslant n} a_i(-\lambda)^i$, where $|V(G)| = n$ and $a_0, ..., a_n \geqslant 0$. Read conjectures:

(4.1) CONJECTURE. *The sequence $a_0, ..., a_n$ is unimodal; that is, for some j,*

$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{j-1} \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_n.$$

Welsh (see [5], page 266) proposed the following strengthening:

(4.2) CONJECTURE. *For $1 \leqslant i \leqslant n-1$, $a_i^2 \geqslant a_{i-1} a_{i+1}$.*

One might hope that our counterexample (2.1) yields a counterexample also to (4.2). But that does not seem to work. It is possible for a polynomial

$$Q(\lambda) = (-1)^n \sum_{0 \leqslant i \leqslant n} a_i(-\lambda)^i$$

to satisfy (4.2) and not to satisfy $Q(\lambda)^2 \geqslant Q(\lambda-1)\, Q(\lambda+1)$. For instance, let

$$Q(\lambda) = \lambda^5 - 8\lambda^4 + 21\lambda^3 - 19\lambda^2 + 5\lambda;$$

then it satisfies (4.2), and yet $Q(0) = 0$, $Q(1) = 0$, $Q(2) = 16$, $Q(3) = 6$, $Q(4) = 36$. It therefore seems likely to me that our counterexample (2.1) will not yield a counterexample to (4.1). It is explained in [2], however, that (2.1) does yield a counterexample to several related conjectures.

Concerning (3.1), one argument for it is the observation from [1] that for any integer $k > 0$,

$$k(1 - P(k-1)/P(k))$$

is the expected number of colours that actually appear in a randomly selected k-colouring of G; and intuitively one might expect this number to be minimum for the graph with no edges. Taking $k = n$, the graph with n vertices and no edges has $P(n)/P(n-1) \simeq e$, which motivated the Bartels–Welsh conjecture. But this intuition is very suspect. For instance, let $n = 100k \log_2 k$ (ignoring the difficulties that these numbers may not be integers) and let G be the complete k-bipartite graph, in which each of the k sets has cardinality $100 \log_2 k$. Then $P(k) = k!$, and

$$P(k+1) \simeq k \cdot 2^{100 \log_2 k - 1}(k+1)! = \tfrac{1}{2}k^{101}(k+1)!$$

and so

$$\frac{P(k+1)}{P(k)} \simeq \frac{1}{2}(k+1)k^{101};$$

while for the graph with n vertices and no edges,

$$\frac{P(k+1)}{P(k)} = \frac{(k+1)^n}{k^n} \simeq k^{100 \log_2(e)}.$$

Thus, for graphs with n vertices, if $k = O(n/\log n)$ then $P(k+1)/P(k)$ is by no means minimized by the graph with no edges.

A related conjecture of Bartels and Welsh, motivated by the same intuition, has already been disproved. Mike Mosca [3] has found two graphs G and H, both with 6 vertices and with G a subgraph of H, so that

$$\frac{P_G(6)}{P_G(5)} > \frac{P_H(6)}{P_H(5)}.$$

(G is obtained from K_5 by splitting one vertex into two vertices of degree 2; and H is obtained from G by joining the two vertices of degree 2.) (1.2) asserts "in the limit" that no such pair G, H exists with $E(G) = \emptyset$.

In the proof of (3.1), we used no properties of graphs except that if we have a colouring of G, and we choose a subset of one colour class and recolour it with some colour that does not yet appear, then this produces another colouring of G. Let us say a set $\mathscr{C}$ of n-colourings of a set V with $|V| = n$ is *consistent* if it has this property; more exactly,

(i) if $\phi \in \mathscr{C}$ and π is a permutation of $\{1, ..., n\}$ then $\psi \in \mathscr{C}$, where ψ is defined by $\psi(v) = \pi(\phi(v))(v \in V)$

(ii) if $\phi \in \mathscr{C}$, $1 \leqslant i \leqslant n$, $X \subseteq \phi^{-1}(i)$, $1 \leqslant j \leqslant n$ and $\phi^{-1}(j) = \emptyset$, then $\psi \in \mathscr{C}$ where

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin X \\ j & \text{if } v \in X. \end{cases}$$

Define $r(\mathscr{C})$ to be the expected number of colours that appear in a randomly (uniformly) selected member of a consistent set $\mathscr{C}$; then the proof of (3.1) shows that $r(\mathscr{C}) \geqslant n(1-\frac{252}{685})$, and so one would expect that $r(\mathscr{C}) \geqslant n(1-(1/e))$. This more general approach has some advantages; for instance, it is easy to reduce the general question to the same question for *symmetric* consistent sets $\mathscr{C}$, those which are invariant under all permutations of V. (Hint: let $v_1, v_2 \in \mathscr{C}_1$ and let $\mathscr{C}'$ be obtained by exchanging v_1, v_2. If the result holds for $\mathscr{C} \cup \mathscr{C}'$ and for $\mathscr{C} \cap \mathscr{C}'$ then it holds for $\mathscr{C}$.) Whether or not a given n-colouring belongs to $\mathscr{C}$ therefore only depends on the *sizes* of its colour classes, and so the problem is reduced to a problem about partitions of an integer. But I don't see how to finish it.

The following seems to be true.

(4.3) CONJECTURE. *For any sequence $s_0, ..., s_n$ of non-negative integers satisfying $\sum s_i = \sum i s_i = n$, there exist i, j with $1 \leqslant i, j$ and $i+j \leqslant n$ so that*

$$n \binom{i+j}{i} s_{i+j} \geqslant e s_i s_j.$$

The proof of (3.1) given earlier was adapted from a proof of (4.3) with e replaced by $\frac{685}{252}$, and it seems likely to me that a proof of (4.3) itself could similarly be adapted to prove (1.2). To see the connection between (4.3) and (3.1), observe that, in the notation of (3.1), we wish to show that

$$\sum_{\mathbf{s} \in \mathscr{S}} \left(\tfrac{252}{685} N(\mathbf{s}) - M(\mathbf{s})\right) \geqslant 0.$$

With e replaced by $\frac{685}{252}$, (4.3) and (3.5) yield that for every $\mathbf{s} \in \mathscr{S}$ there exists $\mathbf{t} \in \mathscr{S}$ differing from $\mathbf{s}$ only a little, with $\frac{252}{685} N(\mathbf{s}) \geqslant M(\mathbf{t})$; and as was explained before, this is the heart of the proof of (3.1).

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